CYCLES FOR THE 3X+1 MAP ON THE GAUSSIAN INTEGERS

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Abstract. The Collatz Conjecture states that for every positive integer n, there exists a positive integer k such that $T^{(k)}(n) = 1$, where $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ by $T(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$ The Finite Cycles Conjecture states that there are only finitely many distinct cycles for T extended to the set of integers. Using Joseph's extension of T to the Gaussian integers, we prove that if $z \in \mathbb{Z}[i]$ is cyclic with extended-T-orbit $\overline{z_0, z_1, ..., z_{n-1}}$ and parity vector $\overline{v_0, v_1, ..., v_{n-1}}$, where $k = |\{j \mid v_j = 1\}|$, $l = |\{j \mid v_j = i\}|$, and $m = |\{j \mid v_j = 1+i\}|$, then

$$\frac{l \ln 3 + (k+m) \ln\left(3 + \frac{1}{\beta}\right)}{\ln 2} \leq n \leq \frac{l \ln 3 + (k+m) \ln\left(3 + \frac{1}{\alpha}\right)}{\ln 2}$$

when $\alpha\beta > 0$, where

$$\alpha \leq \min\{\mathrm{Re}(z_j) \mid 0 \leq j \leq n-1\} \leq \max\{\mathrm{Re}(z_j) \mid 0 \leq j \leq n-1\} \leq \beta.$$

Using this result, we explore the existence of cycles for Joseph's extension of T and determine all cycles having period less than or equal to 400. We also determine all cycles having parity vector of the form $\overline{v_0, v_1, ..., v_{n-1}}$ with $|\{j \mid v_j \neq 0\}| \leq 230$ or $|\{j \mid v_j = 0\}| \leq 131$.

1. Introduction

The $3x + 1$ Conjecture, or Collatz Conjecture, traditionally attributed to Lothar Collatz of the University of Hamburg and dating back to the 1930s, involves the iteration of the function $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$T(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The conjecture states that for every positive integer n, there exists a positive integer k such that $T^{(k)}(n) = 1$.

The *T-orbit* of a point $z_0 \in \mathbb{Z}[i]$ is defined to be the sequence $z_0, z_1, z_2, ...$ where

$$z_j = T(z_{j-1}) \text{ for all } j \geq 1.$$

We say a point z_0 is *cyclic* if $z_0 = z_j$ for some $j \neq 0$ and is *eventually cyclic* if $z_i = z_j$ for some $i \neq j$. If z_0 is cyclic, we denote its T-orbit by $\overline{z_0, z_1, ..., z_{n-1}}$ and say that $\overline{z_0, z_1, ..., z_{n-1}}$ is a *cycle* with *period* or *length* n. We say that two cycles are *equivalent* if the sets containing the elements of their orbits are equal. For example, the cycle $\overline{1, 2}$ is equivalent to $\overline{2, 1}$. In this paper, we will use the term *distinct* to describe a set of cycles in which none of the cycles are equivalent.

The Finite Cycles Conjecture, proposed by several authors, states that there are only finitely many distinct cycles for $T : \mathbb{Z} \to \mathbb{Z}$, and the five distinct cycles starting from $-17, -5, -1, 0$, and 1 are believed to be the only such cycles [3].

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The function T can be extended to the 2-adic integers, $\mathbb{Z}_2$, and further extended to $\mathbb{Z}_2[i]$ given by

$$T(n) = \begin{cases} \frac{n}{2} & \text{if } n \in [0] \\ \frac{3n+1}{2} & \text{if } n \in [1] \\ \frac{3n+i}{2} & \text{if } n \in [i] \\ \frac{3n+1+i}{2} & \text{if } n \in [1+i] \end{cases}$$

where $[x]$ denotes the equivalence class of x in $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i]$ [2]. Notice that $T(\mathbb{Z}[i]) \subseteq \mathbb{Z}[i]$, the Gaussian integers, so that we may consider $T : \mathbb{Z}[i] \to \mathbb{Z}[i]$ as well, as we shall do for the remainder of the paper.

The following information regarding parity vectors may be found in [2].

The *parity vector function for* T, $Q : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]$, is given by the infinite sequence

$$Q(n) = v_0(n), v_1(n), v_2(n), ...$$

where

$$v_j(n) \in \{0, 1, i, 1+i\}$$

and

$$v_j(n) \equiv T^{(j)}(n) \text{ mod } 2 \text{ for all } j \geq 0.$$

(Note that Q is called Q_∞ in [1], [2], and [3].)

Similarly, the *parity vector function of length* k *for* T, $Q_k : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$, is given by the sequence

$$Q_k(n) = v_0(n), v_1(n), v_2(n), ..., v_k(n).$$

Joseph shows that Q_k is periodic with period 2^k and that Q is a bijection. Thus, every $z \in \mathbb{Z}_2[i]$ has a unique parity vector and every element $v \in \mathbb{Z}_2[i]$ is the parity vector of some $z \in \mathbb{Z}_2[i]$, namely $Q^{-1}(v)$. In particular, we define a *periodic parity vector* as the parity vector of a cycle.

It is reasonable to ask if a conjecture analogous to the Finite Cycles Conjecture can be formulated concerning the existence of cycles for T on $\mathbb{Z}[i]$. Determining all cycles for T on $\mathbb{Z}[i]$, or, equivalently, determining which periodic parity vectors are the parity vector of a cycle whose terms are Gaussian integers, would settle the original Finite Cycles Conjecture. Hence, studying the cycles for the extended function can have much to say about the original Collatz problem. By using the main theorem of this paper, we can determine all cycles having period less than or equal to 400 and all cycles having periodic parity vectors $\overline{v_0, v_1, ..., v_{n-1}}$ such that $t_v = |\{0 \leq j \leq n-1 \mid v_j \neq 0\}| \leq 230$ or $n - t_v \leq 131$.

2. Main Results

The proofs of the theorems in this section will be discussed in a subsequent section of the paper.

Let $\overline{z_0, z_1, ..., z_{n-1}}$ be the T-orbit of z. Define $\min_x(z) = \min\{\text{Re}(z_j) \mid 0 \leq j \leq n-1\}$ and $\max_x(z) = \max\{\text{Re}(z_j) \mid 0 \leq j \leq n-1\}$. If $v = \overline{v_0, v_1, ..., v_{n-1}}$ is the periodic parity vector of z, then let $k_v = |\{j \mid v_j = 1\}|$, $l_v = |\{j \mid v_j = i\}|$, $m_v = |\{j \mid v_j = 1+i\}|$, and $t_v = k_v + l_v + m_v$. Thus, t_v represents the number of nonzero entries of v under its overbar. In this paper, we will substitute t, k, l, and m for t_v, k_v, l_v, and m_v, respectively, when v is clear from context.

We state the following theorem, our main result.

Theorem 1. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta > 0$ *and let* $z \in \mathbb{Z}[i]$ *be cyclic and* v *its parity vector. If* $\alpha \leq \min_x(z) \leq \max_x(z) \leq \beta$ *then*

$$\frac{l \ln 3 + (k+m)\ln\left(3+\frac{1}{\beta}\right)}{\ln 2} \leq n \leq \frac{l \ln 3 + (k+m)\ln\left(3+\frac{1}{\alpha}\right)}{\ln 2}.$$

From this, we can prove the following corollary.

Corollary 2. *If* $0 < \alpha < \beta$, *then*

$$\frac{\ln 2}{\ln\left(3+\frac{1}{\alpha}\right)} \leq \frac{t}{n} \leq \frac{\ln 2}{\ln 3}.$$

If $\alpha < \beta < 0$, *then*

$$\frac{\ln 2}{\ln 3} \leq \frac{t}{n} \leq \frac{\ln 2}{\ln\left(3+\frac{1}{\beta}\right)}.$$

Notice that $\frac{t}{n}$ represents the percentage of non-zero elements under the overbar of a periodic parity vector.

A *set theoretic discrete dynamical system* (or simply *dynamical system*, in this paper) is a pair (X, f) where $X \xrightarrow{f} X$. If $X \xrightarrow{f} X$ is a dynamical system, then $S \subseteq X$ is a *subdynamical system* of X if $f(S) \subseteq S$. In this situation, we will write $S \xrightarrow{f} S$.

Two functions $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are said to be *set theoretically conjugate* (or simply *conjugate,* in this paper) if there exists a bijection $X \xrightarrow{h} Y$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes, i.e. $h \circ f = g \circ h$. Conjugacy is an equivalence relation on any set of dynamical systems.

There are several important subdynamical systems of $T : \mathbb{Z}[i] \to \mathbb{Z}[i]$. Let

$$\begin{aligned} W_1 &= \{x+yi \mid 0 < y < x\} \\ W_2 &= \{x+yi \mid 0 < x < y\} \\ W_3 &= \{x+yi \mid y < x < 0\} \\ W_4 &= \{x+yi \mid x < y < 0\}. \end{aligned}$$

Theorem 3. *The following subsets of* $\mathbb{Z}[i]$ *are conjugate subdynamical systems of* $T : \mathbb{Z}[i] \to \mathbb{Z}[i]$*:*
(a) the real axis, the imaginary axis, and the $y = x$ *line.*
(b) the second and fourth quadrants.
(c) W_1 *and* W_2*.*
(d) W_3 *and* W_4*.*

An important property of conjugacy is that it preserves the dynamics of a function. Here, we are concerned mainly with the preservation of cycles by conjugacy. Using this property, we need only examine one set from each of (a)-(d) above in order to determine all cycles. Our work is reduced further by the following two results.

Theorem 4. *The following subsets of* $\mathbb{Z}[i]$ *contain no cyclic points for* $T : \mathbb{Z}[i] \to \mathbb{Z}[i]$*:*
(a) the second and fourth quadrants.
(b) $\{a - i \mid a \leq -2\} \cup \{-1 + ai \mid a \leq -2\}$.

Combining the above results gives an efficient method for determining all cycles of a given period. Theorem $4(a)$ states that there are no cyclic points for $T:\mathbb{Z}[i]\to\mathbb{Z}[i]$ in the second and fourth quadrants. Thus, if we are in search of cycles, we need only look in the first and third quadrants (including the real and imaginary axes). However, by Theorem $3(c)$ and $3(d)$, we have that $\{x+yi \mid y<x\}$ and $\{x+yi \mid x<y\}$ are conjugate subdynamical systems in both the first and third quadrants. By Theorem $3(a)$, we know that the real axis, the imaginary axis, and the line $y=x$ are conjugate subdynamical systems as well. Therefore, we need only look in $\{x+yi \mid 0\le y<x\}$ and $\{x+yi \mid x<y\le 0\}$ in order to find cycles. If we determine that a point $a+bi$ in one of these two sets is cyclic, then we know that the point $b+ai$ is cyclic as well (and if $b=0$, then $a+ai$ is also cyclic). Furthermore, by Theorem $4(b)$, we have that there are no cyclic points for $T:\mathbb{Z}[i]\to\mathbb{Z}[i]$ in the two subsets $\{a-i \mid a\le -2\}$ and $\{-1+ai \mid a\le -2\}$ of the third quadrant.

Given $\alpha\in\mathbb{Z}^{+}$, let $I_\alpha=\left[\frac{\ln 2}{\ln\left(3+\frac{1}{\alpha}\right)},\frac{\ln 2}{\ln 3}\right]$. Note that the interval I_α gets smaller as α gets larger and that $\lim_{\alpha\to\infty}\frac{\ln 2}{\ln\left(3+\frac{1}{\alpha}\right)}=\frac{\ln 2}{\ln 3}$. Let N_α be the least positive integer such that $\frac{j}{N_\alpha}\in I_\alpha$ for some $j\in\mathbb{Z}$. If z is a cyclic point with $\alpha\le\min_x(z)$, $v=\overline{v_0,v_1,\dots,v_{n-1}}$ the parity vector of z, and $t=\left|\{j \mid v_j\neq 0\}\right|$, then by Corollary 2, $\frac{t}{N_\alpha}\in I_\alpha$. Then, $n\ge N_\alpha$ by the definition of N_α. Conversely, if $\operatorname{Re}(z)>0$ and z is cyclic with period $n<N_\alpha$, then $0<\min_x(z)<\alpha$.

Combining the results of the last two paragraphs, we find that in order to determine all cycles in the first quadrant having period less than N_α for a given α, we need only examine the finite set $W_\alpha=\{x+yi \mid 0<x<\alpha \text{ and } 0\le y<x\}$. More specifically, we need only compute the first N_α terms in the T-orbit of all $z\in W_\alpha$ to completely determine all cycles in the first quadrant of period less than N_α.

This procedure for determining cycles is paralleled in the third quadrant. Here, we again use Corollary 2 and let $I_\beta=\left[\frac{\ln 2}{\ln 3},\frac{\ln 2}{\ln\left(3+\frac{1}{\beta}\right)}\right]$. As in the first quadrant, it suffices to examine the finite set $W_\beta=\{x+yi \mid \beta<x<0 \text{ and } x<y\le 0\}$ to determine all cycles in the third quadrant having period less than N_β.

In order to determine all cycles having period less than or equal to 400, we must have N_α and N_β at least 401. The least value of α for which $N_\alpha\ge 401$ is $\alpha=12825$, in which case $N_\alpha=401$. The largest value of β such that $N_\beta\ge 401$ is $\beta=-8461$, in which case $N_\beta=569$. Using a Maple program, we computed the first 401 terms of the T-orbit for all $z\in\{x+yi \mid 0<x<12825 \text{ and } 0\le y<x\}\cup\{x+yi \mid -8461<x<0 \text{ and } x<y\le 0\}$, thus determining all cycles having period less than or equal to 400. Table 1 shows the distribution of all 77 such distinct cycles.

Period	*Number of Cycles*
1	4
2	3
3	9
5	2
8	10
11	5
19	30
46	2
84	10
103	2

Table 1

In order to refer to individual cycles, we define a particular term in the cycle to be its *representative* as follows. For $a+bi,\ c+di \in \mathbb{Z}[i]$ define $a+bi \ll c+di$ if and only if $|a| < |c|$ or $(|a| = |c| \text{ and } |b| < |d|)$. For cycles whose terms are in $W_1 \cup W_4 \cup \mathbb{Z}$, the representative is chosen to be the $\ll$-least term in the cycle. For other cycles, the representative is chosen to be the conjugate image of the representative of the cycle in $W_1 \cup W_4 \cup \mathbb{Z}$ that it corresponds to via the conjugacies of Theorem 3.

Figures 1 through 4 below show the distribution of the representatives of the cycles in $\mathbb{Z}[i]$. Here, the plotted points and numbers indicate the representative elements of each cycle and their periods, respectively.

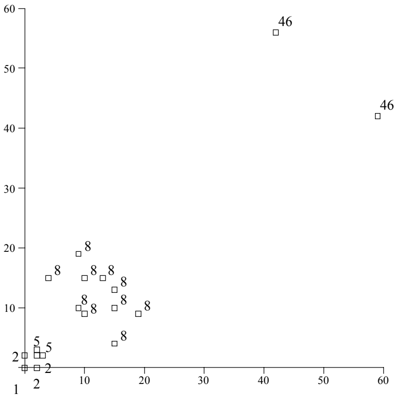

Figure 1: 1st Quadrant

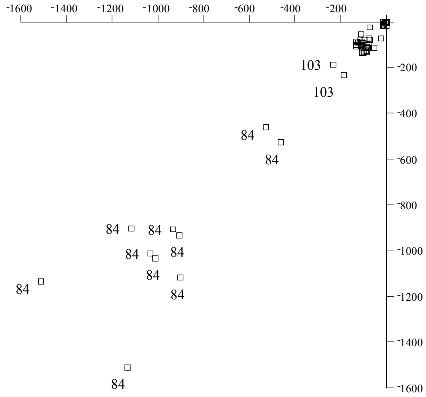

Figure 2: 3rd Quadrant

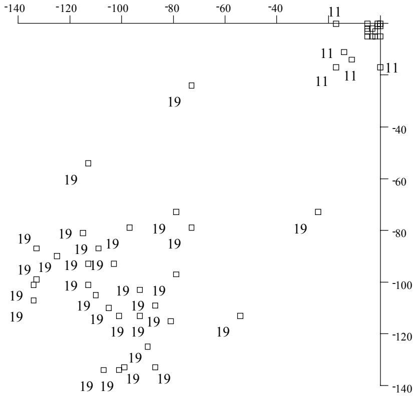

Figure 3: a zoom of Figure 2

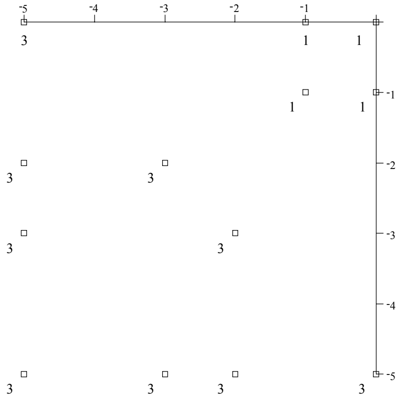

Figure 4: a zoom of Figure 3

In addition to determining cycles of a given period, we can use our main results to determine all cycles having a periodic parity vector of a given composition.

Corollary 5. *Let* z *be cyclic with period* n *and parity vector* v *having either* $t_v \leq 230$ *or* $n - t_v \leq 131$. *Then* $n \leq 400$, *so the cycle containing* z *is listed in Table 2.*

The following table is a list of representative elements of each of the 77 distinct cycles found, the period of each cycle, and t_v where v is the parity vector of the cycle.

Cycle Rep.	*Period*	t_v	*Cycle Rep.*	*Period*	t_v	*Cycle Rep.*	*Period*	t_v
0	1	0	$15+13i$	8	5	$-109-87i$	19	12
$-i$	1	1	$19+9i$	8	5	$-110-105i$	19	12
-1	1	1	$-17i$	11	7	$-113-54i$	19	12
$-1-i$	1	1	$-11-14i$	11	7	$-113-93i$	19	12
i	2	1	$-14-11i$	11	7	$-113-101i$	19	12
1	2	1	-17	11	7	$-115-81i$	19	12
$1+1i$	2	1	$-17-17i$	11	7	$-125-90i$	19	12
$-5i$	3	2	$-24-73i$	19	12	$-133-87i$	19	12
$-2-3i$	3	2	$-54-113i$	19	12	$-133-99i$	19	12
$-2-5i$	3	2	$-73-24i$	19	12	$-134-101i$	19	12
$-3-2i$	3	2	$-73-79i$	19	12	$-134-107i$	19	12
$-3-5i$	3	2	$-79-73i$	19	12	$42+59i$	46	29
-5	3	2	$-79-97i$	19	12	$59+42i$	46	29
$-5-2i$	3	2	$-81-115i$	19	12	$-461-527i$	84	53
$-5-3i$	3	2	$-87-109i$	19	12	$-527-461i$	84	53
$-5-5i$	3	2	$-87-133i$	19	12	$-903-1117i$	84	53
$2+3i$	5	3	$-90-125i$	19	12	$-907-934i$	84	53
$3+2i$	5	3	$-93-103i$	19	12	$-934-907i$	84	53
$4+15i$	8	5	$-93-113i$	19	12	$-1012-1033i$	84	53
$9+10i$	8	5	$-97-79i$	19	12	$-1033-1012i$	84	53
$9+19i$	8	5	$-99-133i$	19	12	$-1117-903i$	84	53
$10+9i$	8	5	$-101-113i$	19	12	$-1134-1511i$	84	53
$10+15i$	8	5	$-101-134i$	19	12	$-1511-1134i$	84	53
$13+15i$	8	5	$-103-93i$	19	12	$-187-234i$	103	65
$15+4i$	8	5	$-105-110i$	19	12	$-234-187i$	103	65
$15+10i$	8	5	$-107-134i$	19	12			

Table 2

3. PROOFS OF MAIN RESULTS

We begin by proving our main result, which we will restate here for convenience.

Theorem 1. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta > 0$ *and let* $z \in \mathbb{Z}[i]$ *be cyclic and* v *its parity vector. If* $\alpha \leq \min_x(z) \leq \max_x(z) \leq \beta$ *then*

$$\frac{l\ln 3 + (k+m)\ln\left(3+\frac{1}{\beta}\right)}{\ln 2} \leq n \leq \frac{l\ln 3 + (k+m)\ln\left(3+\frac{1}{\alpha}\right)}{\ln 2}.$$

Proof. Let $\overline{z_0, z_1, \ldots, z_{n-1}}$ be an n-cycle for T and $v = \overline{v_0, v_1, \ldots, v_{n-1}}$ be its parity vector. Let $0 < \alpha \leq \min_x(z_0)$ and $z_0 = x_0 + y_0 i$.

If $z = x + yi \in Z[i]$, then

$$T(x+yi) = \begin{cases} \frac{1}{2}x + \frac{1}{2}yi & \text{if } x, y \text{ even} \\ \left(\frac{3}{2}x + \frac{1}{2}\right) + \frac{3}{2}yi & \text{if } x \text{ odd, } y \text{ even} \\ \frac{3}{2}x + \left(\frac{3}{2}y + \frac{1}{2}\right) i & \text{if } x \text{ even, } y \text{ odd} \\ \left(\frac{3}{2}x + \frac{1}{2}\right) + \left(\frac{3}{2}y + \frac{1}{2}\right) i & \text{if } x, y \text{ odd.} \end{cases}$$

Define

$$\begin{aligned} T_0(x+yi) &= \frac{1}{2}x + \frac{1}{2}yi \\ T_1(x+yi) &= \left(\frac{3}{2}x + \frac{1}{2}\right) + \frac{3}{2}yi \\ T_i(x+yi) &= \frac{3}{2}x + \left(\frac{3}{2}y + \frac{1}{2}\right) i \\ T_{1+i}(x+yi) &= \left(\frac{3}{2}x + \frac{1}{2}\right) + \left(\frac{3}{2}y + \frac{1}{2}\right) i. \end{aligned}$$

Let $j \in \{0, 1, i, 1+i\}$. Notice that the real parts of $T_j(z)$ depend only on $\mathrm{Re}(z) = x$. Define $R_j(x) = \mathrm{Re}(T_j(x+yi))$ so that:

$R_j(x) = \mathrm{Re}(T_j(x+yi))$
$R_0(x) = \frac{1}{2}x$
$R_1(x) = \frac{3}{2}x + \frac{1}{2}$
$R_i(x) = \frac{3}{2}x$
$R_{1+i}(x) = \frac{3}{2}x + \frac{1}{2}.$

Let $U_j(x) = \omega_j x$ with $\omega_j \in \mathbb{R}$ be an upper bound on $R_j(x)$ for all $x \geq \alpha$. Thus,

Upper bound $U_j(x) = \omega_j x$ on $R_j(x)$
$U_0(x) = \frac{1}{2}x$
$U_1(x) = \omega x$
$U_i(x) = \frac{3}{2}x$
$U_{1+i}(x) = \omega x$

with $\omega = \frac{1}{2}\left(3 + \frac{1}{\alpha}\right)$.

By the definition of cycle, we have

$$z_0 = T_{v_{n-1}} \circ T_{v_{n-2}} \circ \ldots \circ T_{v_1} \circ T_{v_0}(z_0).$$

So,

$$\begin{aligned} x_0 &= \operatorname{Re}(z_0) \\ &= \operatorname{Re}\left(T_{v_{n-1}} \circ T_{v_{n-2}} \circ ... \circ T_{v_1} \circ T_{v_0}(z_0)\right) \\ &= R_{v_{n-1}} \circ R_{v_{n-2}} \circ ... \circ R_{v_1} \circ R_{v_0}(x_0) && \text{by definition of } R_j. \end{aligned}$$

Before we continue, we must prove the following lemma.

Lemma 1. $R_{v_{n-1}} \circ R_{v_{n-2}} \circ ... \circ R_{v_1} \circ R_{v_0}(x_0) \leq U_{v_{n-1}} \circ U_{v_{n-2}} \circ ... \circ U_{v_1} \circ U_{v_0}(x_0)$.

Proof. We will show something stronger, namely

$$R_{v_j} \circ R_{v_{j-1}} \circ ... \circ R_{v_1} \circ R_{v_0}(x_0) \leq U_{v_j} \circ U_{v_{j-1}} \circ ... \circ U_{v_1} \circ U_{v_0}(x_0)$$

for all $0 \leq j \leq n-1$ by finite mathematical induction on j.
Base Case: Let $j = 0$. Then, $R_{v_0}(x_0) \leq U_{v_0}(x_0)$ by construction since $\alpha \leq \min_x(z_0) \leq x_0$.
Inductive Step: Assume

$$R_{v_j} \circ R_{v_{j-1}} \circ ... \circ R_{v_1} \circ R_{v_0}(x_0) \leq U_{v_j} \circ U_{v_{j-1}} \circ ... \circ U_{v_1} \circ U_{v_0}(x_0).$$

Then,

$$\begin{aligned} R_{v_{j+1}} \circ R_{v_j} \circ ... \circ R_{v_1} \circ R_{v_0}(x_0) &= R_{v_{j+1}}\left[R_{v_j} \circ ... \circ R_{v_1} \circ R_{v_0}(x_0)\right] \\ &\leq R_{v_{j+1}}\left[U_{v_j} \circ ... \circ U_{v_1} \circ U_{v_0}(x_0)\right] \\ &\quad \text{(since } R_{v_{j+1}} \text{ is an increasing map)} \\ &\leq U_{v_{j+1}}\left[U_{v_j} \circ ... \circ U_{v_1} \circ U_{v_0}(x_0)\right] \\ &\quad \text{(since } R_{v_{j+1}}(x) \leq U_{v_{j+1}}(x)\text{)} \\ &= U_{v_{j+1}} \circ U_{v_j} \circ ... \circ U_{v_1} \circ U_{v_0}(x_0). \end{aligned}$$

Therefore, by induction,

$$R_{v_{n-1}} \circ R_{v_{n-2}} \circ ... \circ R_{v_1} \circ R_{v_0}(x_0) \leq U_{v_{n-1}} \circ U_{v_{n-2}} \circ ... \circ U_{v_1} \circ U_{v_0}(x_0).$$

∎

Hence, we have

$$\begin{aligned} x_0 &= R_{v_{n-1}} \circ R_{v_{n-2}} \circ ... \circ R_{v_1} \circ R_{v_0}(x_0) \\ &\leq U_{v_{n-1}} \circ U_{v_{n-2}} \circ ... \circ U_{v_1} \circ U_{v_0}(x_0) && \text{by Lemma 1} \\ &= \omega^k \left(\tfrac{3}{2}\right)^l \omega^m \left(\tfrac{1}{2}\right)^{n-k-l-m} x_0 && \text{since } U_{v_i} \circ U_{v_j}(x) = U_{v_j} \circ U_{v_i}(x) \\ &= \tfrac{3^l}{2^n} \cdot \left(3 + \tfrac{1}{\alpha}\right)^{k+m} x_0. \end{aligned}$$

Since $x_0 > 0$, we have

$$1 \leq \frac{3^l}{2^n} \cdot \left(3 + \frac{1}{\alpha}\right)^{k+m}.$$

Solving for n we obtain

$$n \leq \frac{l \ln 3 + (k+m) \ln\left(3 + \frac{1}{\alpha}\right)}{\ln 2}.$$

By a similar argument, if $0 < \max_x(z_0) \leq \beta$, we can show

$$\frac{l \ln 3 + (k+m)\ln\left(3 + \frac{1}{\beta}\right)}{\ln 2} \leq n.$$

Similarly, if $\alpha \leq \min_x(z_0) \leq \max_x(z_0) \leq \beta < 0$, we can show that the bounds on n are the same as those above. Therefore, if $\alpha\beta > 0$, then

$$\frac{l \ln 3 + (k+m)\ln\left(3 + \frac{1}{\beta}\right)}{\ln 2} \leq n \leq \frac{l \ln 3 + (k+m)\ln\left(3 + \frac{1}{\alpha}\right)}{\ln 2}.$$

∎

Using this result, we may now prove the following corollary.

Corollary 2. *If* $0 < \alpha < \beta$, *then*

$$\frac{\ln 2}{\ln\left(3 + \frac{1}{\alpha}\right)} \leq \frac{t}{n} \leq \frac{\ln 2}{\ln 3}.$$

If $\alpha < \beta < 0$, *then*

$$\frac{\ln 2}{\ln 3} \leq \frac{t}{n} \leq \frac{\ln 2}{\ln\left(3 + \frac{1}{\beta}\right)}.$$

Proof. Let $0 < \alpha < \beta$. Then,

$$n \geq \frac{l \ln 3 + (k+m)\ln\left(3 + \frac{1}{\beta}\right)}{\ln 2}$$

by Theorem 1. So,

$$\begin{aligned} n \ln 2 &\geq l \ln 3 + (k+m)\ln\left(3 + \frac{1}{\beta}\right) \\ &\geq l \ln 3 + (k+m)\ln 3 \qquad \text{since } \ln x \text{ is increasing and } \beta > 0 \\ &= t \ln 3. \end{aligned}$$

Therefore,

$$\frac{t}{n} \leq \frac{\ln 2}{\ln 3}.$$

Similarly, from the inequality $n \leq \frac{l \ln 3 + (k+m)\ln\left(3 + \frac{1}{\alpha}\right)}{\ln 2}$ we obtain

$$\frac{\ln 2}{\ln\left(3 + \frac{1}{\alpha}\right)} \leq \frac{t}{n}.$$

We can prove the case $\alpha < \beta < 0$ in a similar manner. ∎

Theorem 3. *The following subsets of* $\mathbb{Z}[i]$ *are conjugate subdynamical systems of* $T : \mathbb{Z}[i] \to \mathbb{Z}[i]$*:*
(a) the real axis, the imaginary axis, and the $y = x$ *line.*
(b) the second and fourth quadrants.
(c) W_1 *and* W_2*.*
(d) W_3 *and* W_4*.*

Proof. Let

$$\begin{aligned} W_5 &= \mathbb{Z} && \text{(the real axis)} \\ W_6 &= \{zi \mid z \in \mathbb{Z}\} && \text{(the imaginary axis)} \\ W_7 &= \{z + zi \mid z \in \mathbb{Z}\} && \text{(the } y = x \text{ line)} \\ W_8 &= \{x + yi \mid x \in \mathbb{Z}^- \text{ and } y \in \mathbb{Z}^+\} && \text{(the } 2nd \text{ quadrant)} \\ W_9 &= \{x + yi \mid x \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z}^-\} && \text{(the } 4th \text{ quadrant).} \end{aligned}$$

Then it is elementary to verify that $T(W_j) \subseteq W_j$ for all j, where $1 \leq j \leq 9$. Therefore, W_j is a subdynamical system of T. Let

$$\begin{aligned} h_1 &: W_5 \to W_6 \text{ by } h_1(z) = zi, \\ h_2 &: W_5 \to W_7 \text{ by } h_2(z) = z + zi, \end{aligned}$$

and for any $x + yi \in \mathbb{Z}[i]$ define $h(x + yi) = y + xi$. Then, it is an elementary, but tedious, exercise to show that h_1, h_2, and the restrictions of h to W_1, W_3, and W_8 are conjugacies with $h(W_1) = W_2$, $h(W_3) = W_4$, and $h(W_8) = W_9$. We omit the proof. ▮

We may now prove our next result.

Theorem 4. *The following subsets of* $\mathbb{Z}[i]$ *contain no cyclic points for* $T : \mathbb{Z}[i] \to \mathbb{Z}[i]$*:*
(a) the second and fourth quadrants.
(b) $\{a - i \mid a \leq -2\} \cup \{-1 + ai \mid a \leq -2\}$.

Proof. *(a)* Let $U = \{x + yi \mid x \in \mathbb{Z}^-,\ y \in \mathbb{Z}^+\}$ and $V = \{x + yi \mid x \in \mathbb{Z}^+,\ y \in \mathbb{Z}^-\}$. Let $x+yi \in V$. Let $r = |x + yi|$ so that $x + yi = re^{i \arctan\left(\frac{y}{x}\right)}$. Then,

$$T\left(re^{i \arctan\left(\frac{y}{x}\right)}\right) = \begin{cases} r_1 e^{i \arctan\left(\frac{y}{x}\right)} & \text{if } x, y \text{ are even} \\ r_2 e^{i \arctan\left(\frac{3y}{3x+1}\right)} & \text{if } x \text{ odd, } y \text{ even} \\ r_3 e^{i \arctan\left(\frac{3y+1}{3x}\right)} & \text{if } x \text{ even, } y \text{ odd} \\ r_4 e^{i \arctan\left(\frac{3y+1}{3x+1}\right)} & \text{if } x, y \text{ odd} \end{cases}$$

for some r_1, r_2, r_3, and r_4 in $\mathbb{R}^+$.

If x and y are both even, then $\arg(x + yi) = \arg(T(x + yi))$. If x and y are not both even, then $\arg(T(x + yi)) = \arctan\left(\frac{3y+\varepsilon_2}{3x+\varepsilon_1}\right)$ for some $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ with $\varepsilon_1 + \varepsilon_2 > 0$. Then,

$$\begin{aligned} & y < x \\ \Longrightarrow\ & \varepsilon_1 y < \varepsilon_2 x \\ \Longrightarrow\ & 3xy + \varepsilon_1 y < 3xy + \varepsilon_2 x \\ \Longrightarrow\ & \frac{y}{x} < \frac{3y + \varepsilon_2}{3x + \varepsilon_1}. \end{aligned}$$

Therefore, $\arctan\left(\frac{y}{x}\right) < \arctan\left(\frac{3y+\varepsilon_2}{3x+\varepsilon_1}\right)$ since arctan is an increasing map. This implies that $\arg(x + yi) < \arg(T(x + yi))$ for all $x + yi$ such that x and y are not both even. Therefore, if $z \in V$, then $\arg(z) = \arg(T(z))$ if $z \equiv 0 \bmod 2$ and $\arg(z) < \arg(T(z))$ otherwise. By mathematical induction on n, we see that if $z \in V$, then $\arg(z) \leq \arg\left(T^{(n)}(z)\right)$ for all n.

Assume $z \in V$ is cyclic of period n. Thus, $n \geq 1$. Since $z \neq 0$, the parity vector of z is not $\overline{0}$. Therefore, there exists a j such that $T^{(j)}(z) \not\equiv 0 \bmod 2$. So,

$$\begin{array}{rcll} \arg\left(T^{(j)}(z)\right) & = & \arg\left(T^{(j+n)}(z)\right) & \\ & = & \arg\left(T^{((n-1)+(j+1))}(z)\right) & \\ & = & \arg\left(T^{(n-1)}\left(T^{(j+1)}(z)\right)\right) & \\ & \geq & \arg\left(T^{(j+1)}(z)\right) & \text{since } \arg(z) \leq \arg\left(T^{(n)}(z)\right) \\ & > & \arg\left(T^{(j)}(z)\right) & \text{since } \arg(T(z)) > \arg(z) \text{ when } z \not\equiv 0 \bmod 2 \end{array}$$

which is a contradiction.

Therefore, z is not cyclic for all $z \in V$. Since U and V are conjugate subdynamical systems of $T : \mathbb{Z}[i] \to \mathbb{Z}[i]$ by Theorem 3(b), then z is not cyclic for all $z \in U$. Therefore, there are no cyclic points for T in the second and fourth quadrants.

(b) Assume $a - i$ is cyclic. Then, $T^{(n)}(a-i) = a - i$ for some $n \geq 1$. If a is even, then $T(a-i) = \frac{3}{2}a - i$. Since $a < 0$, $1 < \frac{3}{2}$ implies $\frac{3}{2}a < a$. Therefore, $\operatorname{Re}(T(a-i)) < \operatorname{Re}(a-i)$ when a is even. If a is odd, then $T(a-i) = \frac{3a+1}{2} - i$. Since $a < -1$, we have $1 < -a \Rightarrow 1 < 2a - 3a \Rightarrow 3a + 1 < 2a \Rightarrow \frac{3a+1}{2} < a$. Therefore, $\operatorname{Re}(T(a-i)) < \operatorname{Re}(a-i)$ when a is odd. Thus, for all $a \leq -2$, $\operatorname{Re}(T(a-i)) < \operatorname{Re}(a-i)$ while $\operatorname{Im}(T(a-i)) = \operatorname{Im}(a-i)$. Hence, there is no n such that $T^{(n)}(a-i) = a - i$ and thus $a - i$ is not cyclic for $a \leq -2$. By a similar argument (or by the conjugacy $h(x+yi) = y + xi$), there are no cyclic points in $\{-1 + ai \mid a \leq -2\}$. ∎

We can use Corollary 2 to prove the following result.

Corollary 5. *Let z be cyclic with period n and parity vector v having either $t_v \leq 230$ or $n - t_v \leq 131$. Then $n \leq 400$, so the cycle containing z is listed in Table 2.*

Proof. Let $0 < \alpha \leq \min_x(z) \leq \max_x(z) \leq \beta$. It is easy to verify by direct calculation that the orbit of z enters a cycle listed in Table 2 for all $z \in W_1 \cup \mathbb{Z}^+$ such that $\operatorname{Re}(z) < 3$. Thus we let $\alpha = 3$. Hence, by Corollary 2, $\frac{\ln 2}{\ln\left(\frac{10}{3}\right)} \leq \frac{t}{n}$, so

$$n \leq \frac{\ln\left(\frac{10}{3}\right)}{\ln 2} t.$$

Also by Corollary 2, $\frac{t}{n} \leq \frac{\ln 2}{\ln 3}$, so $1 - \frac{\ln 2}{\ln 3} \leq 1 - \frac{t}{n}$ and therefore

$$n \leq \frac{\ln 3}{\ln 3 - \ln 2}(n - t).$$

Now let $\alpha \leq \min_x(z) \leq \max_x(z) \leq \beta < 0$. It is easy to verify by direct calculation that the orbit of z either enters a cycle listed in Table 2 or is divergent by Theorem 4(b) for all $z \in W_4 \cup \mathbb{Z}^-$ such that $\operatorname{Re}(z) > -5$. Thus we let $\beta = -5$. Hence, by Corollary 2, $\frac{\ln 2}{\ln 3} \leq \frac{t}{n}$, so

$$n \leq \frac{\ln 3}{\ln 2} t.$$

Also by Corollary 2, $\frac{t}{n} \leq \frac{\ln 2}{\ln\left(3 + \frac{1}{\beta}\right)}$, so $1 - \frac{\ln 2}{\ln 2.8} \leq 1 - \frac{t}{n}$ and therefore

$$n \leq \frac{\ln 2.8}{\ln 2.8 - \ln 2}(n - t).$$

Let $z \in W_1 \cup W_4 \cup \mathbb{Z}$ be cyclic with period n. If $-5 < \min_x(z) < 3$ then it is easy to verify by direct calculation that the cycle containing z is listed in Table 2. If $\min_x(z)$ is not between -5 and 3, then the inequalities above hold. In particular, either $n \leq \frac{\ln 3}{\ln 2} t$ or $n \leq \frac{\ln\left(\frac{10}{3}\right)}{\ln 2} t$, so in either case

$n \leq \frac{\ln\left(\frac{10}{3}\right)}{\ln 2} t$. Hence, $t \leq 230$ implies $n < 400$. Similarly, $n \leq \frac{\ln 3}{\ln 3 - \ln 2}(n-t)$ or $n \leq \frac{\ln 2.8}{\ln 2.8 - \ln 2}(n-t)$, so in either case $n \leq \frac{\ln 2.8}{\ln 2.8 - \ln 2}(n-t)$. Thus, $n - t \leq 131$ implies $n \leq 400$. Therefore, in every case, if $t \leq 230$ or $n - t \leq 131$, then the cycle containing z is listed in Table 2. ∎

The approach of [1] for determining cycles utilized the parity vector function Q associated with $T : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$. The authors determined all integer cycles having periodic parity vector of the form $v = \overline{\underbrace{000...0}_{a\ zeros}1}$ by proving that $Q^{-1}(v)$ was not an integer for $a > 1$. It is reasonable to ask if the method and results of [1] can be extended to $T : \mathbb{Z}[i] \rightarrow \mathbb{Z}[i]$.

It is an elementary exercise to extend their method to $T : \mathbb{Z}[i] \rightarrow \mathbb{Z}[i]$ and in turn determine all cycles having periodic parity vector v, with $t_v = 1$. If the method of [1] could be generalized to arbitrary periodic parity vectors, then it would settle the Finite Cycles Conjecture. Using this method, we can, in fact, determine all cycles having periodic parity vector v, with $t_v \leq 3$. However, this task becomes increasingly complicated as t_v increases. To illustrate, in order to determine all cycles having periodic parity vector of the form $v = \overline{\underbrace{000...0}_{a\ zeros}p\underbrace{000...0}_{b\ zeros}q\underbrace{000...0}_{c\ zeros}r}$ where $p, q, r \neq 0$ by the method of [1], we first compute

$$Q^{-1}(v) = \frac{2^a\left(9\varepsilon_1 + 3 \cdot 2^{b+1}\varepsilon_3 + 2^{b+c+2}\varepsilon_5\right)}{2^{a+b+c+3} - 27} + \frac{2^a\left(9\varepsilon_2 + 3 \cdot 2^{b+1}\varepsilon_4 + 2^{b+c+2}\varepsilon_6\right)}{2^{a+b+c+3} - 27} i$$

where $\varepsilon_1 + \varepsilon_2 i = p$, $\varepsilon_3 + \varepsilon_4 i = q$, and $\varepsilon_5 + \varepsilon_6 i = r$. Then, we must determine which cases of a, b, c, and ε_j, with $1 \leq j \leq 6$, imply $Q^{-1}(v) \in \mathbb{Z}[i]$. This method seems to become intractably hard as t_v increases.

However, Theorem 1 and Corollary 2 are far more efficient for determining cycles for a given t_v. This efficiency is best illustrated by Corollary 5, resulting from our main Theorem and Corollary, which is used to determine all cycles having periodic parity vector v such that $t_v \leq 230$ or $n - t_v \leq 131$. This is clearly a vast improvement over the results obtained by the method discussed in the previous paragraph, which only determined all cycles having periodic parity vector v such that $t_v \leq 3$.

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